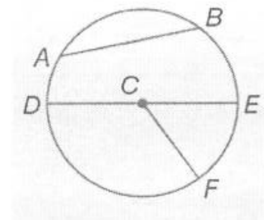
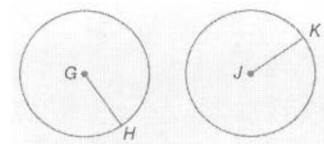
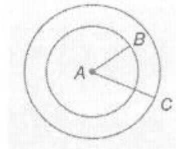
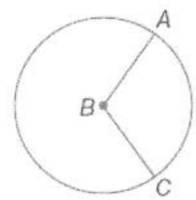
CIRCLES, ARCS, ANGLES, AND CHORDS [1]

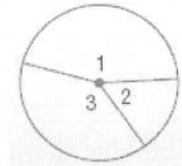
*A **circle** is the locus or set of all points in a plane equidistant from a given point called the **center** of the circle.*



Words	*Figure*
Two circles are **congruent circles** if and only if they have congruent radii.	
Concentric Circles are coplanar circles that have the same center.	

*A **central angle** of a circle is an angle with a vertex in the center of the circle. Its sides contain two radii of the circle.*

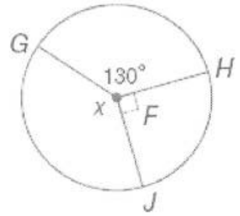


Sum of Central Angles	
Words	*Example*
The sum of the measures of the central angles of a circle with no interior points in common is 360 degrees. $m\angle 1 + m\angle 2 + m\angle 3 = 360°$	

Examples

Find the value of x.

a.

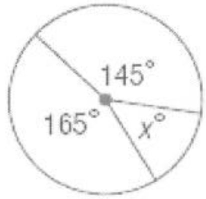


$x + 130 + 90° = 360°$

$220 + x = 360$

$x = 140°$

b.

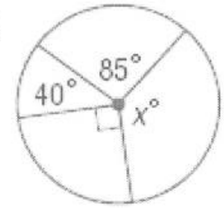


$x + 145 + 165 = 360$

$x + 310 = 360$

$x = 50°$

c.

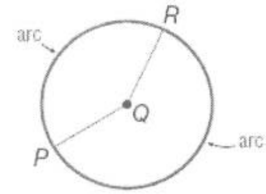


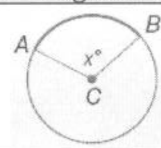
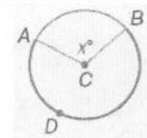
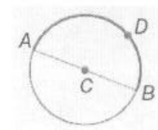
$x + 90 + 40 + 85 = 360°$

$x + 215 = 360$

$x = 145°$

*An **arc** is a portion of a circle defined by two endpoints. A central angle separates the circle into two arcs with measures related to the measure of the central angle.*



Arc	*Figure*
Minor Arc is the shortest arc connecting two endpoints on a circle	 $m\widehat{AB} = x^\circ$
Major Arc is the longest arc connecting two endpoints on a circle	 $m\widehat{ADB} = 360 - x$
Semicircle is an arc with endpoints that lie on a diameter	 $m\widehat{ADB} = 180^\circ$

Examples

$\overline{GJ}$ is a diameter of circle K. Identify each arc as a major arc, minor arc, or semicircle. Then find its measure.

a. $m\widehat{GH}$

minor arc

$= 122^\circ$

b. $m\widehat{GLH}$

major arc

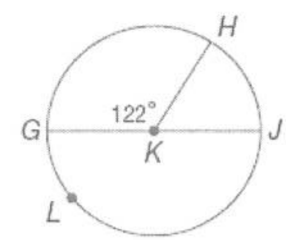
$= 360 - 122$

$= 238^\circ$

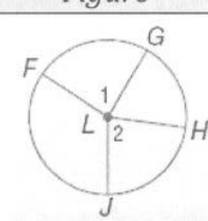
c. $m\widehat{GLJ}$

semi circle

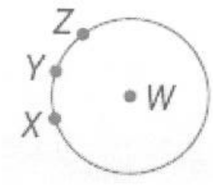
$= 180^\circ$



***Congruent arcs** are arcs in the same or congruent circles that have the same measure.*

Words	*Example*	*Figure*
In the same circle or in congruent circles, two minor arcs are congruent if and only if their central angles are congruent.	$\widehat{FG} \cong \widehat{JH}$ iff $\angle 1 \cong \angle 2$	

***Adjacent arcs** are arcs that have exactly one point in common.*

Arc Addition Postulate		
Words	*Example*	*Figure*
The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.	$m\widehat{XZ} = m\widehat{XY} + m\widehat{YZ}$	

Examples

Find each measure in circle F.

a. $m\widehat{AED}$

$= m\widehat{AE} + m\widehat{ED}$

$= 63 + 90$

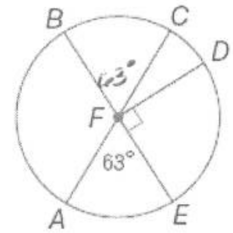
$= 153°$

b. $m\widehat{ADB}$

$= m\widehat{ADC} + m\widehat{BC}$

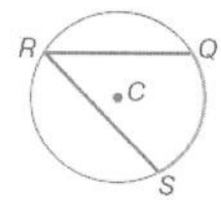
$= 180 + 63$

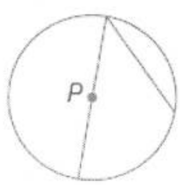
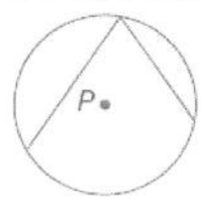
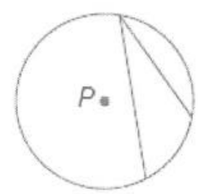
$= 243°$

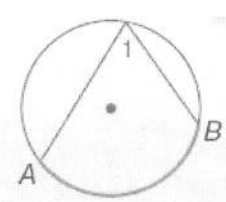


INSCRIBED ANGLES [2]

*An **inscribed angle** has a vertex on a circle and sides that contain chords of the circle. An **intercepted arc** has endpoints on the sides of an inscribed angle and lies in the interior of the inscribed angle.*



Case 1	*Case 2*	*Case 3*
		

Inscribed Angle Theorem		
Words	*Example*	*Figure*
If an angle is inscribed in a circle, then the measure of the angle equals one half the measure of its intercepted arc.	$\cancel{m\widehat{AB} = \frac{1}{2} m\angle 1}$ $m\angle 1 = \frac{1}{2} m\widehat{AB}$	

Examples

Find each measure.

a. $m\angle P$

$= \frac{1}{2} m\widehat{MN}$

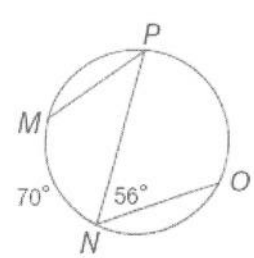
$= \frac{1}{2}(70)$

$= 35°$

b. $m\widehat{PO}$ $= 2\, m\angle N$

$= 2(56)$

$= 112°$



c. $m\widehat{CF}$

$= 2\,m\angle D$

$= 2(40°)$

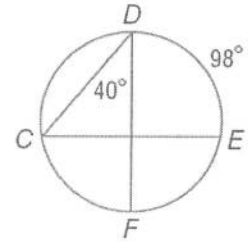
$= 80°$

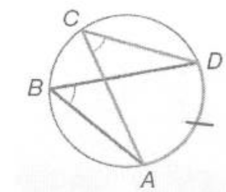
d. $m\angle C$

$= \frac{1}{2}\,m\widehat{DE}$

$= \frac{1}{2}(98)$

$= 49°$



Words	Example	Figure
If two inscribed angles of a circle intercept the same arc or congruent arcs, then the angles are congruent.	$\angle C \cong \angle B$ since $m\angle B = \frac{1}{2}\,m\widehat{AD}$ $m\angle C = \frac{1}{2}\,m\widehat{AD}$	

Examples

a. Find $m\angle T$

$m\angle T = m\angle U$

$3x - 5 = 2x + 15$

$x = 20$

$m\angle T = 3x - 5$

$= 3(20) - 5$

$= 55°$

b. If $m\angle S = 3x$ and $m\angle V = x + 16$, find $m\angle S$.

$m\angle S = m\angle V$

$3x = x + 16$

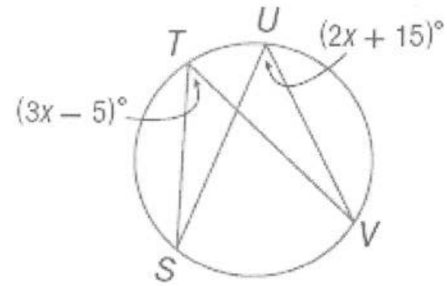
$2x = 16$

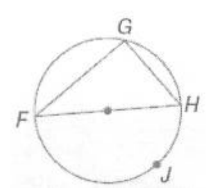
$x = 8$

$m\angle S = 3x$

$= 3(8)$

$= 24$



Thales' Theorem		
Words	**Examples**	**Figures**
An inscribed angle of a triangle intercepts a diameter or semicircle if and only if the angle is a right angle.	$\angle G$ is a right $\angle$ iff FH is a diameter	

Examples

1. a. Find $m\angle F$.

$m\angle G = 90°$

$4x + 2 + 9x - 3 + 90 = 180$

$13x + 89 = 180$

$13x = 91$

$x = 7$

$m\angle F = 4(7) + 2$

$= 30°$

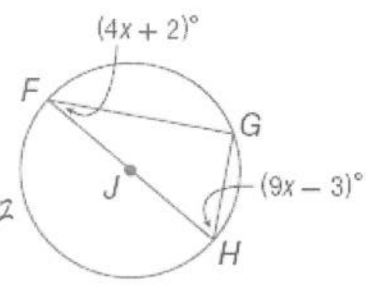
b. If $m\angle F = 7x + 2$ and $m\angle H = 17x - 8$, find x.

$7x + 2 + 17x - 8 + 90 = 180$

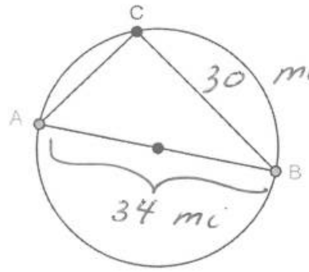
$24x + 84 = 180$

$24x = 96$

$x = 4$



2. In the figure below, $\overline{AB}$ is the diameter of a circle of radius 17 miles. If $BC = 30$ miles, what is AC?



$a^2 + b^2 = c^2$

$30^2 + (AC)^2 = 34^2$

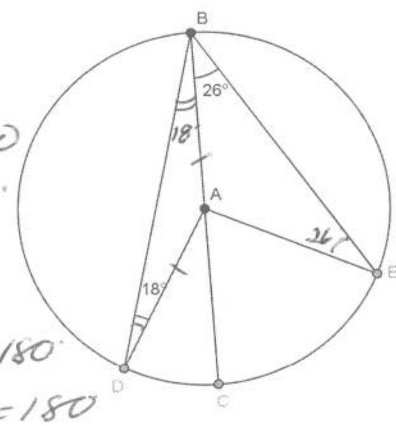
$900 + (AC)^2 = 1156$

$(AC)^2 = 256$

$AC = \sqrt{256}$

$AC = 16$

3. In the circle shown, $\overline{BC}$ is a diameter with center A.



a. Find $m\angle DAB$.

$m\angle ABD = 18°$
if 2 sides of a Δ are $\cong$ then their opp. $\angle$s are $\cong$

$18 + 18 + m\angle DAB = 180°$
$36 + m\angle DAB = 180$
$m\angle DAB = 144°$

b. Find $m\angle BAE$.

$m\angle E = 26°$
if 2 sides of a Δ are $\cong$ then their opp. $\angle$'s are $\cong$

$26 + 26 + m\angle BAE = 180°$
$52 + m\angle BAE = 180$
$m\angle BAE = 128°$

c. Find $m\angle DAE$.

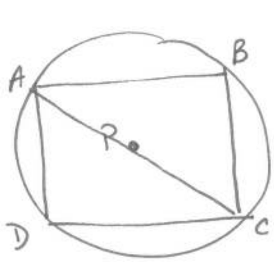
$m\angle DAE = m\angle DAC + m\angle EAC$

$m\angle DAE = 36 + 52$
$= 88°$

$m\angle DAC = 180 - 144$
$= 36°$

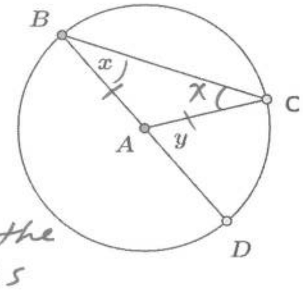
$m\angle EAC = 180 - 128$
$= 52°$

4. Rectangle $ABCD$ is inscribed in circle P. Boris says that diagonal AC could pass through the center, but it does not have to pass through the center. Is Boris correct? Explain your answer in words, or draw a picture to help you explain your thinking.



Boris is incorrect. $\overline{AC}$ does have to pass through the center. $\angle B$ is a right $\angle$ since a rectangle has 4 rt. $\angle$s. By Thales' theorem if an inscribed $\angle$ of a circle is a rt. $\angle$ then it intercepts a diameter of the circle. Since a diameter must pass through the center, $\overline{AC}$ must intersect P.

5. Prove that $y = 2x$ in the diagram provided.



$\overline{AB} \cong \overline{AC}$ — All radii of a circle are $\cong$

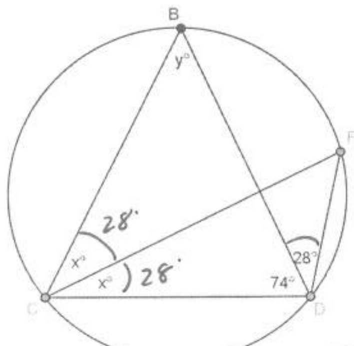
$m\angle C = x$ — if 2 sides of a $\triangle$ are $\cong$ then their opp $\angle$s are $=$ in measure

~~m∠~~ $y = x + x$ — an exterior $\angle$ of a $\triangle$ ='s the sum of its remote int. $\angle$s

$y = 2x$ — simplification

6. Find the measures of the labeled angles.

a.



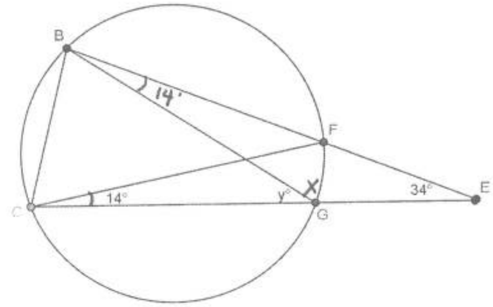
$x = 28°$ — In a circle, inscribed $\angle$s that share the same intercepted arc are $\cong$

$y + 74 + 28 + 28 = 180$ — the sum of the int. $\angle$s of a $\triangle$ is $180°$

$y + 130 = 180$

$y = 50°$

b.



$x + 14 + 34 = 180$ — the sum of the int. $\angle$s of a $\triangle$ is $180°$

$x + 48 = 180$

$x = 132°$

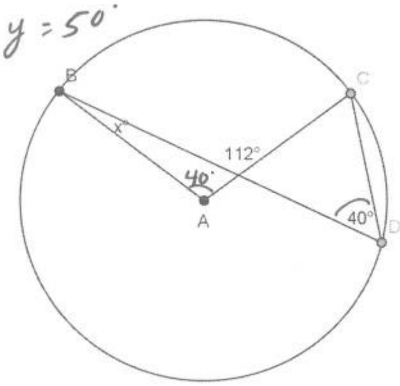
$y = 180 - 132$ or -

$y = 48°$

$y = 14 + 34$

$y = 48$ — An ext. $\angle$ of a $\triangle$ =s the sum of its remote int. $\angle$s.

c.

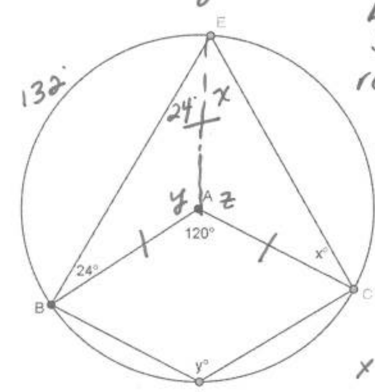


$112 = x + 40$

$x = 72°$

An ext. $\angle$ of a $\triangle$ =s the sum of its remote int. $\angle$s.

d.



$x + x + 108 = 180$

$2x = 72$

$x = 36°$

$y + 24 + 24 = 180$

$y + 48 = 180$

$y = 132°$

$132 + 120 + z = 360°$

$252 + z = 360$

$z = 108$

$y = \frac{1}{2}(132 + 108)$

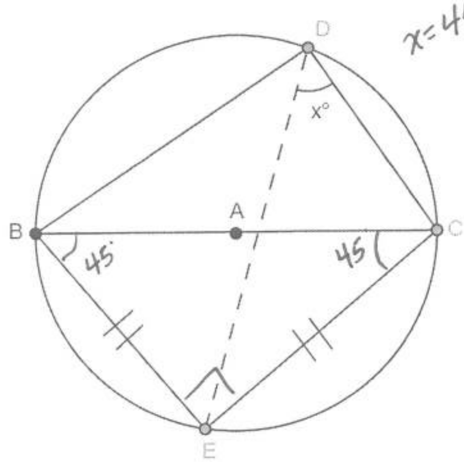
$= \frac{1}{2}(240)$

$y = 120°$

Find the value x in each figure below, and describe how you arrived at the answer.

1. Hint: Thales' theorem

$x = 45°$



∡BEC is a rt ∡. — if an inscribed ∡ of a circle intercepts a diameter then it is a rt ∡.

m∡CBE = m∡BCE — if 2 sides of a △ are ≅ then their opp. ∡s are ≅

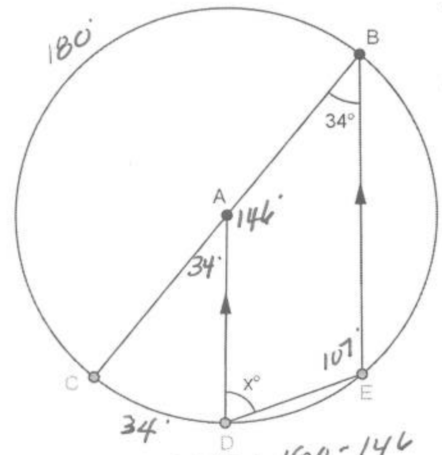
m∡CBE = m∡BCE = 45° — the sum of the int. ∡s of a △ is 180°

m∡x = 45° — if 2 inscribed ∡s of a circle share the same intercepted arc, then they are ≅

2.

m∡BAD = 180 − 34
= 146°

if 2 ∥ lines are cut by a transv. consec. int. ∡s are supp.

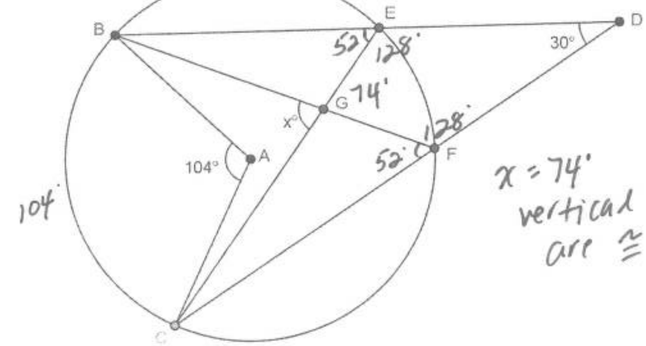


m∡CAD = 180 − 146
= 34° — 2 ∡'s that form a linear pair are supp.

m∡E = ½(180+34)
= ½(214)
= 107 — an inscribed ∡ of a circle is half the measure of its intercepted arc.

X + 107 = 180
X = 73° — if 2 ∥ lines are cut by a transv. then consec. int. ∡s are supp.

3.



$x = 74°$ vertical ∡s are ≅

m∡BEG = m∡CFG = 52° an inscribed ∡ of a circle is half the measure of its intercepted arc.

m∡DEG = 180 − 52°
= 128° — ~~the sum~~ 2 ∡s that form a linear pair are supp.

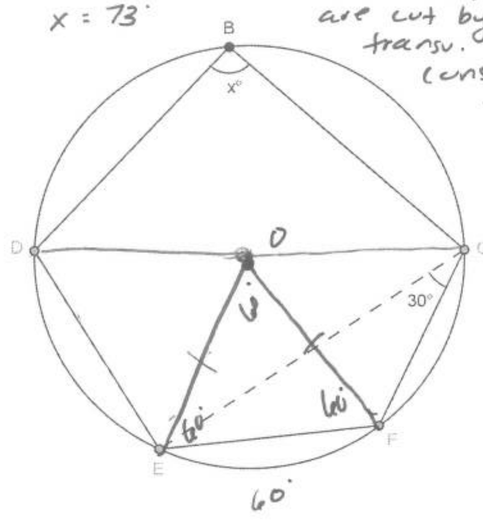
m∡DFG = 128°

x+ m∡EGF + 128 + 128 + 30 = 360° — the sum of the int. ∡s of a quad is 360°

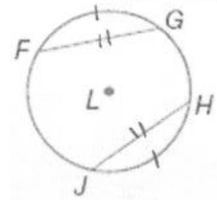
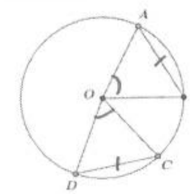
m∡EGF + 286 = 360

m∡EGF = 74°

4.

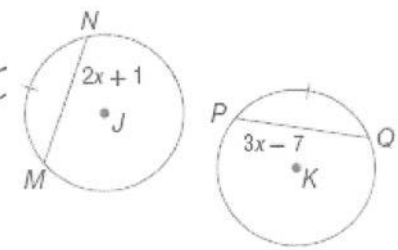


ARCS AND CHORDS [4]

Words	Example
In the same circle or in congruent circles, two minor arcs are congruent if and only if their corresponding chords are congruent $\overset{\frown}{FG} \cong \overset{\frown}{JH}$ iff $\overline{FG} \cong \overline{JH}$	
Arcs between parallel chords are congruent. if $\overline{FG} \parallel \overline{JH}$ then $\overset{\frown}{HG} \cong \overset{\frown}{JF}$	
In a circle, central angles are congruent if and only if their corresponding chords are congruent. $\angle AOB \cong \angle DOC$ iff $\overline{AB} \cong \overline{DC}$	

Examples

a. In the figures, circle J ≅ circle K and $\overset{\frown}{MN} \cong \overset{\frown}{PQ}$. Find PQ.



$\overline{MN} \cong \overline{PQ}$ if 2 arcs in ≅ circles are ≅, then their corresp. chords are ≅.

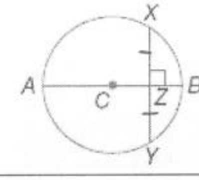
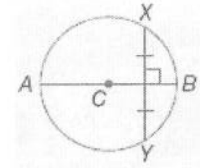
$2x+1 = 3x-7$

$x = 8$

$PQ = 3x-7$
$= 3(8)-7$
$= 24-7$

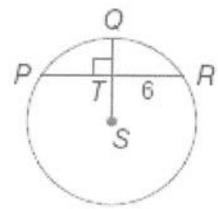
$PQ = 17$

If a line, segment, or ray divides an arc into two congruent arcs, then it ***bisects*** *the arc.*

Bisecting Arcs and Chords		
Words	*Examples*	*Figures*
If a diameter (or radius) of a circle is perpendicular to a chord, then it bisects the chord and its arc.	If $\overline{AB}$ is a diameter and $\overline{AB} \perp \overline{XY}$ then $\overline{AB}$ bisects $\overline{XY}$ and $\overset{\frown}{XY}$	
If a diameter (or radius) of a circle bisects a chord, then it must be perpendicular to the chord.	If $\overline{AB}$ is a diameter and $\overline{AB}$ bisects $\overline{XY}$, then $\overline{AB} \perp \overline{XY}$	

Examples

a. In circle S, $m\widehat{PQR} = 98$. Find $m\widehat{PQ}$.



Since $\overline{QS}$ ~~bisect~~ is a radius and ~~bi~~ is $\perp$ to $\overline{PR}$, $\overline{QS}$ bisects $\overline{PR}$ and $\widehat{PQR}$.

$m\widehat{PQ} = \frac{1}{2} m\widehat{PQR}$

$= \frac{1}{2}(98)$

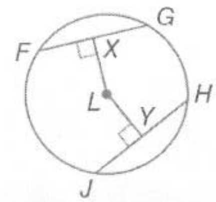
$m\widehat{PQ} = 49$

b. In circle S, find *PR*.

$PR = 2(RT)$

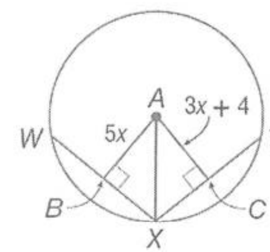
$PR = 2(6)$

$PR = 12$

Words	*Example*	*Figure*
In the same circle or in congruent circles, two chords are congruent if and only if they are equidistant from the center.	~~XL~~ $\overline{FG} \cong \overline{JH}$ iff $\overline{XL} \cong \overline{YL}$ (& $\overline{XL} \perp \overline{FG}$, $\overline{YL} \perp \overline{JH}$)	

Examples

1. In circle A, $WX = XY = 22$. Find *AB*.



$AB = AC$

$5x = 3x + 4$

In a circle, if 2 chords are $\cong$, then they are equidistant from the center.

$2x = 4$

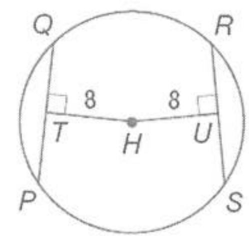
$x = 2$

$AB = 5x$

$AB = 5(2)$

$AB = 10$

2. In circle H, $PQ = 3x - 4$ and $RS = 14$. Find *x*.



$PQ = RS$

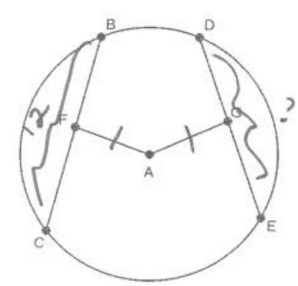
$3x - 4 = 14$

$3x = 18$

$x = 6$

In a circle, if 2 chords are equidistant from the center, then they are $\cong$.

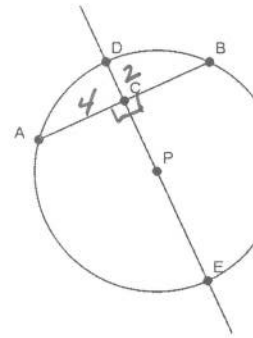
3. Given circle *A* shown, $AF = AG$ and $BC = 22$. Find *DE*.



$DE = BC = 12$

In a circle if 2 chords are equidistant from the center, then they are $\cong$.

4. In the figure, circle P has a radius of 10. $\overline{AB} \perp \overline{DE}$.



   a. If $AB = 8$, what is the length of AC?

$AC = \frac{1}{2}(8)$

$AC = 4$

If the diameter of a circle is $\perp$ to a chord, then it bisects the chord.

   b. If $DC = 2$, what is the length of AB?

radius

$PC = PD - DC$

$PC = 10 - 2$

$PC = 8$

$(PC)^2 + (BC)^2 = (BP)^2$

$8^2 + (BC)^2 = 10^2$

$64 + (BC)^2 = 100$

$(BC)^2 = 36$

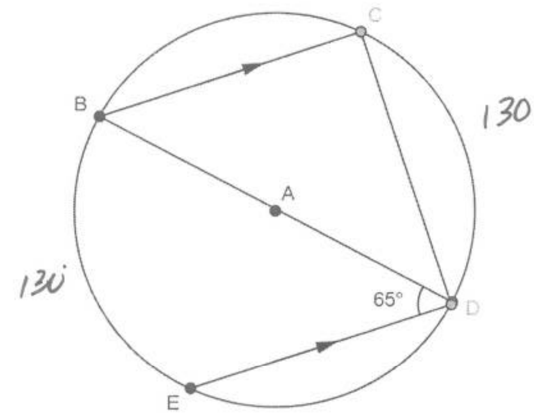
$BC = 6$

$AB = 2(BC)$

$= 2(6)$

$AB = 12$

5. Find the angle measure of $\widehat{CD}$ and $\widehat{ED}$.



$m\widehat{BE} = 2(m\angle EDB)$

$= 2(65)$

$= 130°$

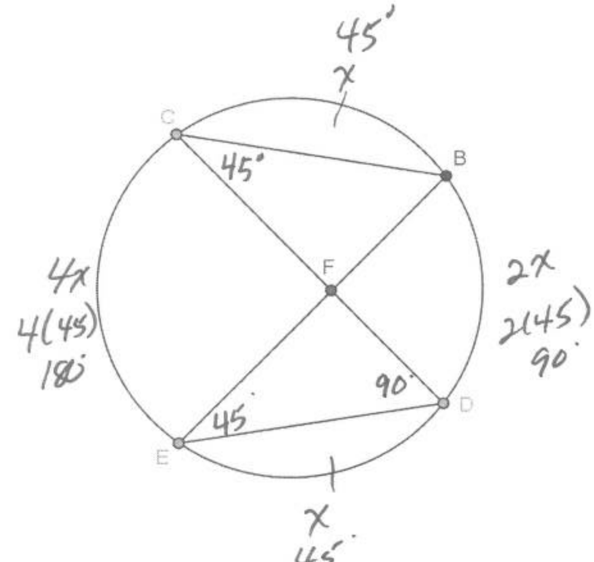
$m\widehat{CD} = m\widehat{EB} = 130°$

$m\widehat{ED} + m\widehat{BE} = 180$

$m\widehat{ED} + 130 = 180$

$m\widehat{ED} = 50°$

6. $m\widehat{CB} = m\widehat{ED}$ and $m\widehat{BC} : m\widehat{BD} : m\widehat{EC} = 1:2:4$. Find



   a. $m\angle BCF$

$m\angle BCF = \frac{1}{2} m\widehat{BD}$

$= \frac{1}{2}(90°)$

$= 45°$

   b. $m\angle EDF$

$m\angle EDF = \frac{1}{2} m\widehat{EC}$

$= \frac{1}{2}(180)$

$= 90°$

   c. $m\angle CFE$

$m\angle CFE = m\angle D + m\angle E$

$= 90° + 45°$

$m\angle CFE = 135°$

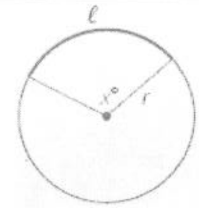
$4x + x + x + 2x = 360°$

$8x = 360°$

$x = 45°$

ARC LENGTH AND AREA OF SECTORS [5]

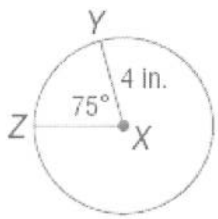
***Arc length** is the distance between the endpoints along an arc measured in linear units. Since an arc is a portion of a circle, its length is a fraction of the circumference.*

Arc Length		
Words	*Proportion*	*Figure*
The ratio of the length of an arc L to the circumference of the circle is equal to the ratio of the degree measure of the arc to 360.	$\frac{l}{2\pi r} = \frac{x^\circ}{360^\circ}$	

Examples

Find the length of $\widehat{ZY}$. Round to the nearest hundredth.

a.



$$\frac{l}{2\pi r} = \frac{x}{360^\circ}$$

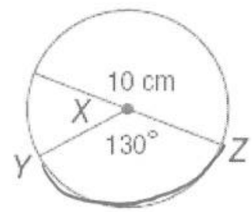
$$\frac{l}{2\pi(4)} = \frac{75}{360}$$

$$\frac{360l}{360} = \frac{600\pi}{360}$$

$$l = 5.23599$$

$$l \approx 5.24 \text{ in}$$

b.



$$\frac{l}{2\pi r} = \frac{x}{360^\circ}$$

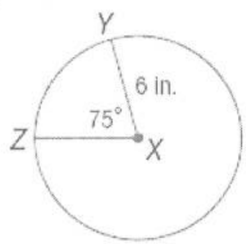
$$\frac{l}{2\pi(5)} = \frac{130^\circ}{360^\circ}$$

$$\frac{360l}{360} = \frac{1300\pi}{360}$$

$$l = 11.3446$$

$$l \approx 11.34$$

c.



$$\frac{l}{2\pi r} = \frac{x}{360^\circ}$$

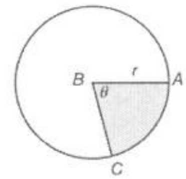
$$\frac{l}{2\pi(6)} = \frac{75}{360^\circ}$$

$$\frac{360l}{360} = \frac{900\pi}{360}$$

$$l = 7.85398$$

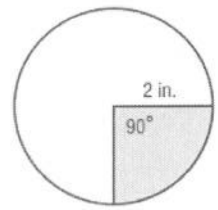
$$l \approx 7.85$$

*The area of a circle is found by using the formula $A = \pi r^2$. A **sector** is a pie-shaped portion of the circle enclosed by 2 radii and the edge of the circle. A central angle of a sector is an angle whose vertex is at the center of the circle and crosses the circle.*

Area of Sectors		
Words	*Proportion*	*Figure*
The area of a sector θ is proportional to the part that the central angle is of 360°.	$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{360}$	

Examples

a. Find the area of the sector shown at the right.



$\frac{A}{\pi(2^2)} = \frac{90}{360}$ $\qquad \frac{360A}{360} = \frac{360\pi}{360}$

$A = \pi \text{ in}^2$

b. Find the area of a sector if the circle has a radius of 10 centimeters and the central angle measures 72.

$\frac{A}{\pi(10^2)} = \frac{72}{360}$ $\qquad \frac{360A}{360} = \frac{7200\pi}{360}$

$A = 62.8319 \text{ cm}^2$

c. Find the area of a sector if the circle has a radius of 5 inches and the central angle measures 60.

$\frac{A}{\pi(5^2)} = \frac{60}{360}$ $\qquad \frac{360A}{360} = \frac{1500\pi}{360}$

$A = 13.09 \text{ in}^2$

d. If the area of a sector is 15π square centimeters and the radius of the circle is 5 centimeters, find the measure of the central angle.

$\frac{15\pi}{\pi(5^2)} = \frac{x}{360}$ $\qquad \frac{5400\pi}{25\pi} = \frac{25\pi x}{25\pi}$

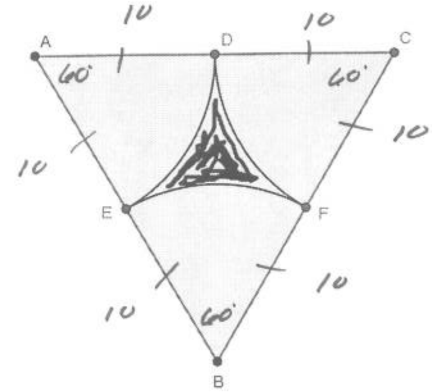
$x = 216°$

e. Find the measure of the central angle that intercepts a sector that is $\frac{1}{3}$ the area of the circle.

$\frac{\frac{1}{3}A}{A} = \frac{x}{360}$ $\qquad \frac{120A}{A} = \frac{Ax}{A}$

$x = 120°$

1. ΔABC is an equilateral triangle with edge length 20 cm. D, E, and F are midpoints of the sides. The vertices of the triangle are the centers of the circles creating the arcs shown. Find the following (round to the nearest hundredth):



a. The area of the sector with center A.

$$\frac{A}{\pi(10^2)} = \frac{60}{360} \qquad \frac{360A}{360} = \frac{6000\pi}{360}$$

$$A = 52.3599$$

$$A \approx 52.36 \text{ cm}^2$$

b. The area of triangle ABC.

$$K = \tfrac{1}{2}ab\sin C$$

$$K = \tfrac{1}{2}(20)(20)\sin 60°$$

$$K = 200\left(\tfrac{\sqrt{3}}{2}\right) \qquad K = 100\sqrt{3} \text{ cm}^2$$

c. The area of the shaded region.

$$= 100\sqrt{3} - 3(52.3599)$$

$$= 16.1254 \text{ cm}^2 \qquad \approx 16.13 \text{ cm}^2$$

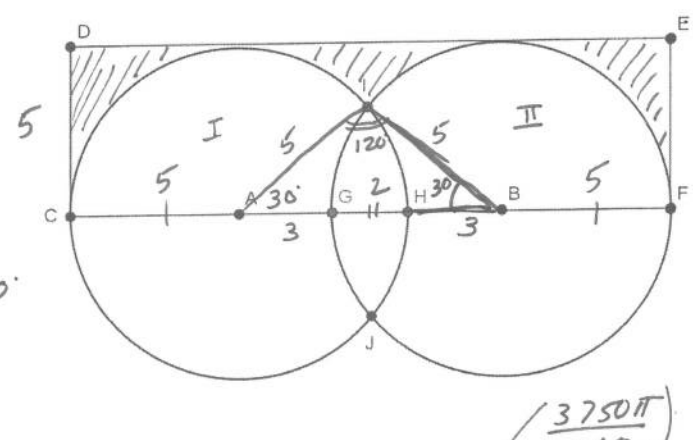
d. The perimeter of the shaded region.

$$\frac{\ell}{2\pi r} = \frac{x}{360°} \qquad \frac{360\ell}{360} = \frac{1200\pi}{360} \qquad P = 3(10.472)$$

$$\frac{\ell}{2\pi(10)} = \frac{60}{360} \qquad \ell = 10.472 \qquad P \approx 31.4159 \text{ cm}$$

$$P \approx 31.42 \text{ cm}$$

2. In the figure shown, $AC = BF = 5$ cm, $GH = 2$ cm, and $m\angle HBI = 30°$. Find the area in the rectangle, but outside of the circles (round to the nearest hundredth).



$$A_{\square} = bh$$

$$A_{\square} = 18(5)$$

$$A_{\square} = 90 \text{ cm}^2$$

$$A_{\Delta} = \tfrac{1}{2}ab\sin C$$

$$= \tfrac{1}{2}(5)(5)\sin 120°$$

$$= 12.5\sin 120°$$

$$\frac{A_I}{\pi(5^2)} = \frac{150}{360°}$$

$$360A_I = 3750\pi$$

$$A_I = 32.749 = A_{II}$$

$$A_{\square} = 90 - 12.5\sin 120° - 2\left(\cancel{32.749}\ \frac{3750\pi}{360}\right)$$

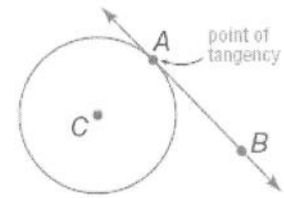
$$= \cancel{13.6767} \text{ cm}^2$$

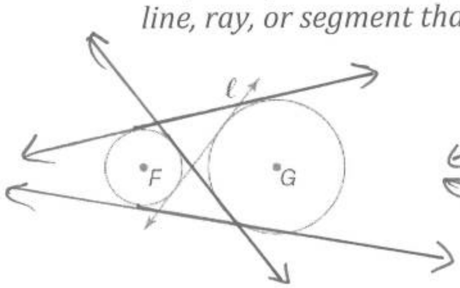
$$= 13.7248$$

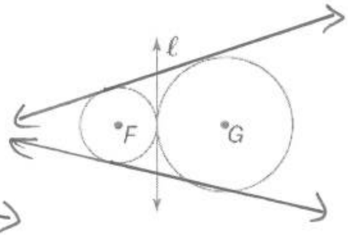
$$\approx 13.72 \text{ cm}^2$$

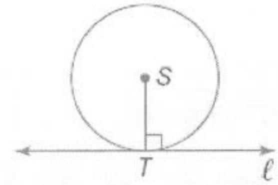
TANGENTS [6]

*A **tangent** is a line in the same plane as a circle that intersects the circle in exactly one point, called the **point of tangency**. A **common tangent** is a line, ray, or segment that is tangent to two circles in the same plane.*







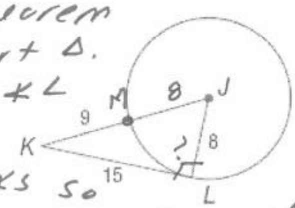
Theorem		
Words	*Example*	*Figure*
In a plane, a line is tangent to a circle if and only if it is perpendicular to a radius drawn to the point of tangency.	ℓ is a tangent iff $\overline{ST} \perp$ line ℓ and $\overline{ST}$ is a radius.	

Examples

a. $\overline{JL}$ is a radius of circle *J*. Determine whether $\overline{KL}$ is tangent to circle *J*. Justify your answer.

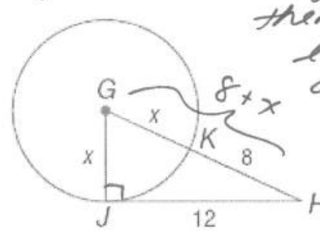
$a^2+b^2=c^2$
$8^2+15^2=17^2$
$64+225=289$
$289=289$
$\overline{KL}$ is a tangent.

Since the pythag. theorem is true, ΔJKL is a rt Δ. A rt Δ has a rt ∡ so ∡L is a rt. ∡. ⊥ lines intersect forming rt. ∡s so $JL \perp \overline{KL}$. If a line is ⊥ to the radius of a circle at its point of tangency, then the line is a tangent.

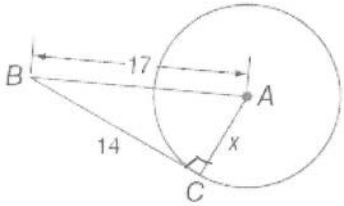


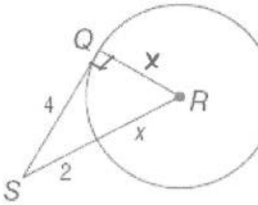
b. $\overline{JH}$ is tangent to circle *G* at *J*. Find the value of *x*.

$a^2+b^2=c^2$
$x^2+12^2=(x+8)^2$
$x^2+144=x^2+16x+64$
$80=16x$
$x=5$



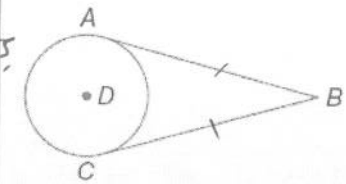
c. Find the value of *x*. Assume that segments that appear to be tangent are tangent.





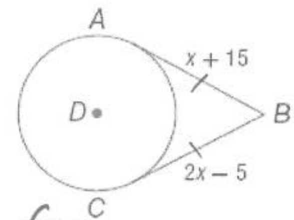
$x^2+14^2=17^2$
$x^2+196=289$
$x^2=93$
$x=\sqrt{93}$

$4^2+x^2=(x+2)^2$
$16+x^2=x^2+4x+4$
$12=4x$
$x=3$

Theorem		
Words	*Example*	*Figure*
If two segments from the same exterior point are tangent to a circle, then they are congruent.	If $\overline{AB}$ and $\overline{BC}$ are both tangents, then $\overline{AB} \cong \overline{CB}$	

Examples

a. $\overline{AB}$ and $\overline{CB}$ are tangent to circle D. Find the value of x.



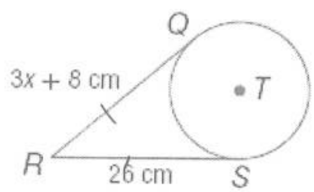
$\overline{AB} \cong \overline{CB}$

$x+15=2x-5$

$20=x$

if 2 segments are tangent to the same circle from the same ext. pt, then they are $\cong$.

b. Find the value of x. Assume that segments that appear to be tangent are tangent.

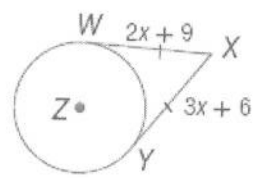


$\overline{RQ} \cong \overline{RS}$

$3x+8=26$

$3x=18$

$x=6$

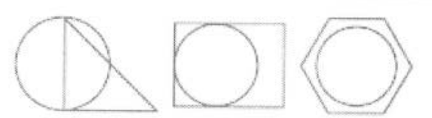


$\overline{WX} \cong \overline{YX}$

$2x+9=3x+6$

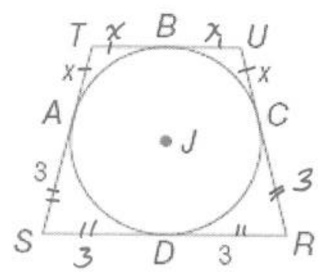
$3=x$

Circumscribed Polygons

Circumscribed Polygons	Polygons Not Circumscribed
	

Examples

a. Quadrilateral $RSTU$ is circumscribed about circle J. If the perimeter is 18 units, find x.



$18=x+x+x+x+3+3+3+3$

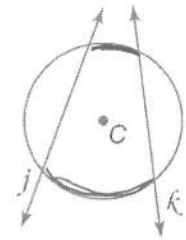
$18=4x+12$

$6=4x$

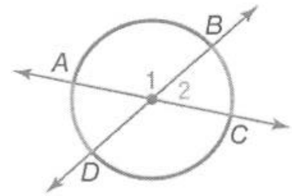
$x=\frac{6}{4}$

$x=1.5$

SECANTS, TANGENTS, AND ANGLE MEASURES [7]



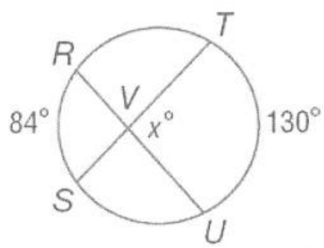
*A **secant** is a line that intersects a circle in exactly two points. When two secants intersect inside a circle, the angles formed are related to the arcs they intercept.*

Theorem		
Words	*Example*	*Figure*
If two secants or chords intersect in the interior of a circle, then the measure of an angle formed is one half the sum of the measure of the arcs intercepted by the angle and its vertical angle.	$m\angle 1 = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$ $m\angle 2 = \frac{1}{2}(m\widehat{BC} + m\widehat{AD})$	

Examples

Find x.

a.



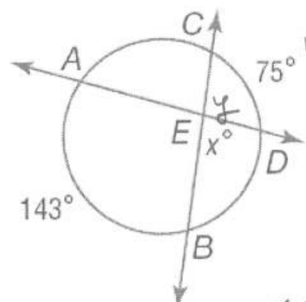
$m\angle TVU = \frac{1}{2}(m\widehat{TU} + m\widehat{RS})$

$x = \frac{1}{2}(130 + 84)$

$x = \frac{1}{2}(214)$

$x = 107°$

b.



$m\angle DEB = \frac{1}{2}(m\widehat{BD} + m\widehat{AC})$

we don't know these

$m\angle CED = \frac{1}{2}(m\widehat{CD} + m\widehat{AB})$

$y = \frac{1}{2}(75 + 143)$

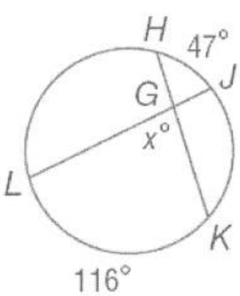
$y = \frac{1}{2}(218)$

$y = 109$

$x = 180 - 109$

$x = 71°$

c.



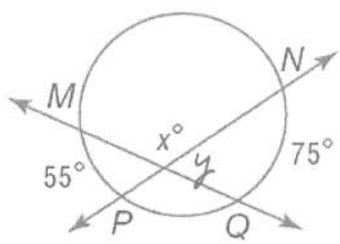
$m\angle LGK = \frac{1}{2}(m\widehat{LK} + m\widehat{JH})$

$x = \frac{1}{2}(116 + 47)$

$x = \frac{1}{2}(163)$

$x = 81.5°$

d.



$y = \frac{1}{2}(55 + 75)$

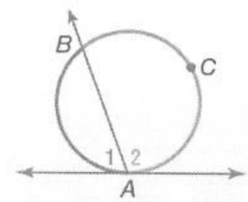
$y = \frac{1}{2}(130)$

$y = 65$

$x = 180 - y$

$x = 180 - 65$

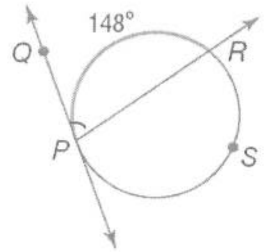
$x = 115°$

Theorem		
Words	*Example*	*Figure*
If a secant and a tangent intersect at the point of tangency, then the measure of each angle formed is one half the measure of its intercepted arc.	$m\angle 1 = \frac{1}{2} m\widehat{AB}$ $m\angle 2 = \frac{1}{2} m\widehat{ACB}$	

Examples

Find each measure.

a. $m\angle QPR$

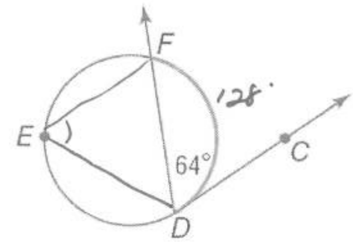


$m\angle QPR = \frac{1}{2} m\widehat{PR}$

$= \frac{1}{2}(148)$

$= 74°$

b. $m\angle DEF$



$m\angle FDC = \frac{1}{2} m\widehat{DF}$

$64 = \frac{1}{2} m\widehat{DF}$

$m\widehat{DF} = 128°$

$m\angle E = \frac{1}{2} m\widehat{DF}$

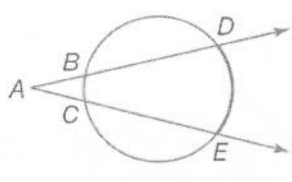
$= \frac{1}{2}(128)$

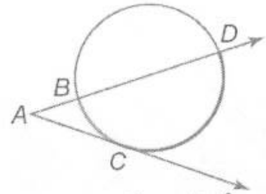
$m\angle DEF = 64°$

Theorem
If two secants, a secant and a tangent, or two tangents intersect in the exterior of a circle, then the measure of the angle formed is half the difference of the measures of the intercepted arcs.

Examples

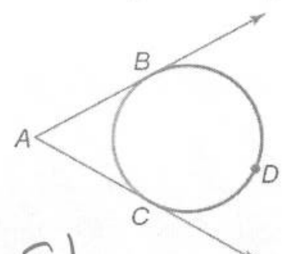
$m\angle A = \frac{1}{2}(m\widehat{DE} - m\widehat{BC})$





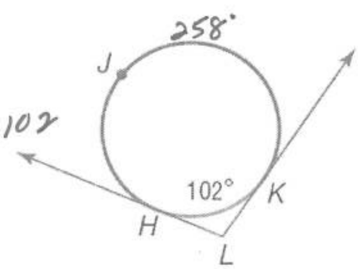
$m\angle A = \frac{1}{2}(m\widehat{CD} - m\widehat{BC})$

$m\angle A = \frac{1}{2}(m\widehat{BDC} - m\widehat{BC})$



Find each measure.

a. $m\angle L$



$m\widehat{HJK} = 360 - 102$

$m\widehat{HJK} = 258°$

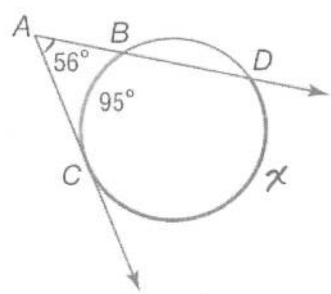
$m\angle L = \frac{1}{2}(m\widehat{HJK} - m\widehat{HK})$

$m\angle L = \frac{1}{2}(258° - 102°)$

$m\angle L = \frac{1}{2}(156)$

$m\angle L = 78°$

b. $m\widehat{CD}$



$m\angle A = \frac{1}{2}(m\widehat{CD} - m\widehat{BC})$

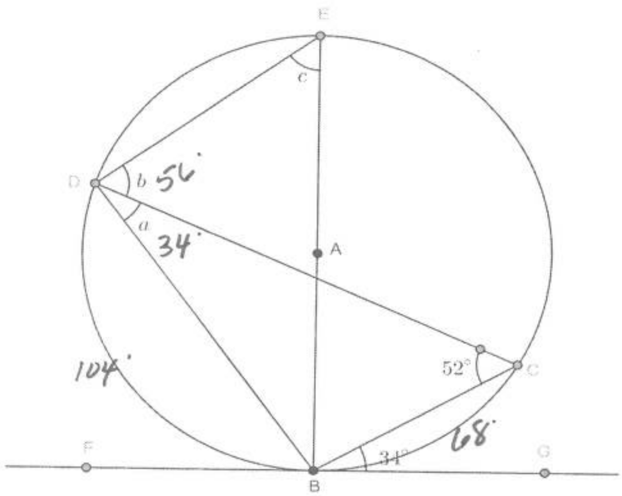
$2 \cdot 56° = \frac{1}{2}(x - 95°) \cdot 2$

$112 = x - 95$

$x = 207°$

Find $x, y, a, b,$ and/or c.

1.



$m\widehat{BC} = 2\, m\angle CBG$
$= 2(34)$
$= 68°$

$m\angle$
$a = \frac{1}{2}\, m\widehat{BC}$
$= \frac{1}{2}(68)$
$= 34°$

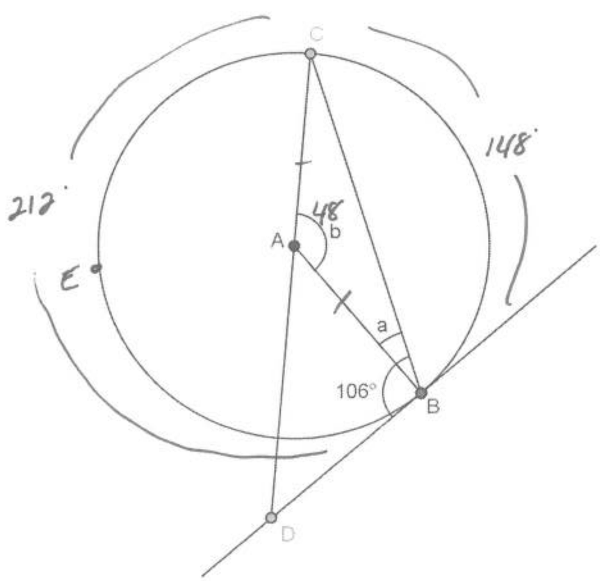
$m\angle EDB = \frac{1}{2}\, m\widehat{ECB}$
$= \frac{1}{2}(180)$
$= 90°$

$a + b = 90$
$34 + b = 90$
$b = 56°$

$m\widehat{BD} = 2\, m\angle C$
$= 2(52)$
$= 104°$

$c = \frac{1}{2}\, m\widehat{BD}$
$c = \frac{1}{2}(104)$
$c = 52°$

2.



$m\angle CBD = \frac{1}{2}\, m\widehat{CEB}$
$106 = \frac{1}{2}\, m\widehat{CEB}$
$m\widehat{CEB} = 212°$

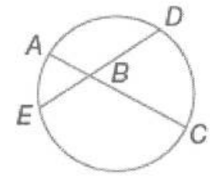
$m\widehat{CB} = 360 - m\widehat{CEB}$
$= 360° - 212°$
$= 148°$

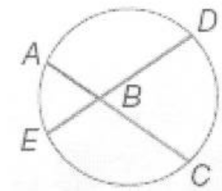
$m\angle CAB = m\widehat{CB}$
$b = 148°$

$m\angle C + m\angle CBA + b = 180°$
$a + a + 48 = 180$
$2a = 132°$
$a = 66°$

MEASURES OF TANGENT SEGMENTS, SECANT SEGMENTS, AND CHORDS [8]

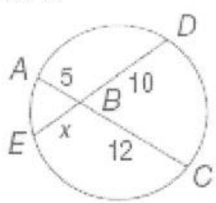
When two chords intersect inside a circle, each chord is divided into two segments, called ***chord segments****. In the figure, chord $\overline{AC}$ is divided into segments $\overline{AB}$ and $\overline{BC}$. Likewise, chord $\overline{ED}$ is divided into segments $\overline{EB}$ and $\overline{BD}$.*



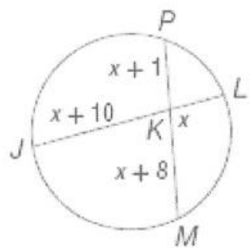
Theorem		
Words	*Example*	*Figure*
If two chords intersect in a circle, then the products of the lengths of the chord segments are equal.	$AB \cdot BC = EB \cdot BD$	

Examples

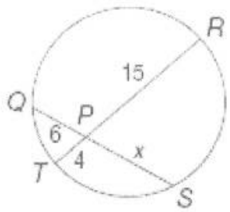
Find x.



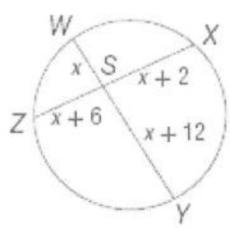
$AB \cdot BC = EB \cdot BD$
$5(12) = x(10)$
$60 = 10x$
$x = 6$



$PK \cdot KM = JK \cdot KL$
$(x+1)(x+8) = (x+10)x$
$\cancel{x^2} + 9x + 8 = \cancel{x^2} + 10x$
$x = 8$

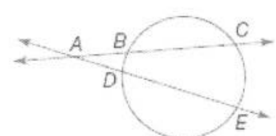


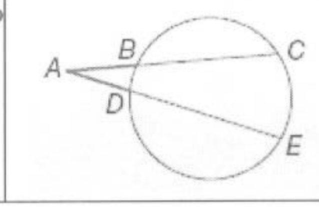
$QP \cdot PS = TP \cdot PR$
$6x = 4(15)$
$6x = 60$
$x = 10$



$WS \cdot SY = ZS \cdot SX$
$x(x+12) = (x+6)(x+2)$
$\cancel{x^2} + 12x = \cancel{x^2} + 8x + 12$
$4x = 12$
$x = 3$

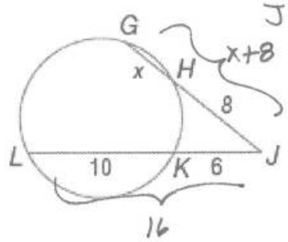
A ***secant segment*** *is a segment of a secant line that has exactly one endpoint on the circle. A secant segment that lies in the exterior of the circle is called an* ***external secant segment****.*



Theorem		
Words	*Example*	*Figure*
If two secants intersect in the exterior of a circle, then the product of the measures of one secant segment and its external secant segment is equal to the product of the measures of the other secant and its external secant segment.	$AC \cdot AB = AE \cdot AD$	

Examples

Find x.



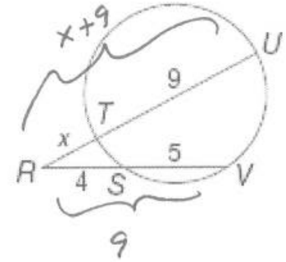
$JG \cdot JH = JL \cdot JK$

$(x+8)8 = 16(6)$

$8x + 64 = 96$

$8x = 32$

$x = 4$



$RU \cdot RT = RV \cdot RS$

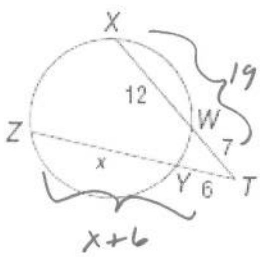
$x(x+9) = 4(9)$

$x^2 + 9x = 36$

$x^2 + 9x - 36 = 0$

$(x+12)(x-3) = 0$

$x = \cancel{-12} \quad x = 3$



$TX \cdot TW = TZ \cdot TY$

$19(7) = (x+6)6$

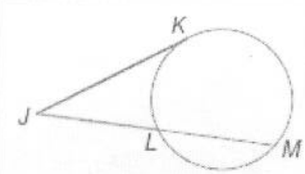
$133 = 6x + 36$

$97 = 6x$

$\cancel{x = 49.5}$

$x = \frac{97}{6}$

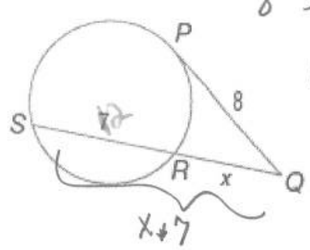
An equation similar to the one above can be used when a secant and a tangent intersect outside a circle. In this case, the ***tangent segment****, or segment of a tangent with one endpoint on the circle, is both the exterior and whole segment.*

Theorem		
Words	*Example*	*Figure*
If a tangent and a secant intersect in the exterior of a circle, then the square of the measure of the tangent is equal to the product of the measures of the secant and its external secant segment.	$(JK)^2 = JM \cdot JL$	

*

Examples

Find x.



$(PQ)^2 = QS(QR)$

$8^2 = (x+7)x$

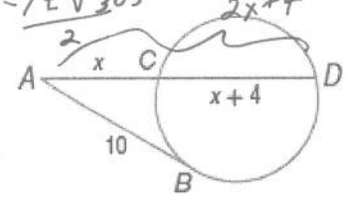
$64 = x^2 + 7x$

$0 = x^2 + 7x - 64$

$x = \frac{-7 \pm \sqrt{49 - 4(1)(-64)}}{2(1)}$

$x = \frac{-7 \pm \sqrt{305}}{2}$

$x = -7 + \sqrt{305}$



$(AB)^2 = AD(AC)$

$10^2 = (2x+4)x$

$100 = 2x^2 + 4x$

$0 = 2x^2 + 4x - 100$

$0 = 2(x^2 + 2x - 50)$

$x = \frac{-2 \pm \sqrt{4 - 4(1)(-50)}}{2(1)}$

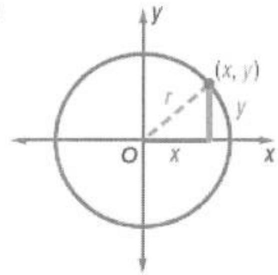
$x = -2 \pm \sqrt{4+200}$ [illegible] $x = -2 + \sqrt{204}$

$x = \frac{-2 \pm 2\sqrt{51}}{2}$

$x = -1 + \sqrt{51}$

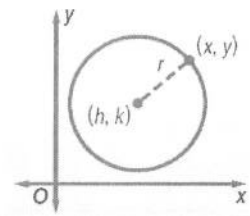
EQUATIONS OF CIRCLES (STANDARD FORM) [9

Since all points on a circle are equidistant from the center, you can find an equation of a circle by using the Distance Formula.



$x^2 + y^2 = r^2$

<table>
<tr><th colspan="2">Equation of a Circle – Center-Radius/Standard Form</th></tr>
<tr><th>Words</th><th>Figure</th></tr>
<tr><td>The standard form of the equation of a circle with center (h, k) and radius r, is $(x - h)^2 + (y - k)^2 = r^2$.

(Center-Radius Form)</td><td></td></tr>
</table>

Examples

Write the equation of each circle.

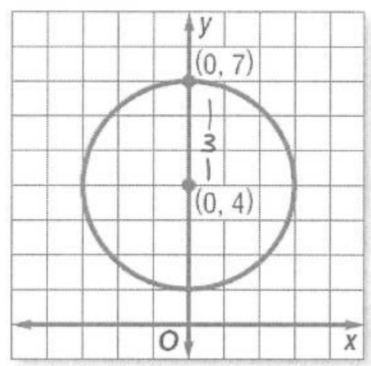
a. center at (1, -8), radius 7

$(x-h)^2 + (y-k)^2 = r^2$

$(x-1)^2 + (y+8)^2 = 7^2$

$(x-1)^2 + (y+8)^2 = 49$

b. the circle graphed at the right



$(x-h)^2 + (y-k)^2 = r^2$

$(x-0)^2 + (y-4)^2 = 3^2$

$x^2 + (y-4)^2 = 9$

c. center at origin, radius $\sqrt{10}$

$x^2 + y^2 = r^2$

$x^2 + y^2 = \sqrt{10}^2$

$x^2 + y^2 = 10$

d. Center at (4, -1), diameter 8

$r = 4$

$(x-h)^2 + (y-k)^2 = r^2$

$(x-4)^2 + (y+1)^2 = 4^2$

$(x-4)^2 + (y+1)^2 = 16$

Examples

Write the equation of each circle.

a. center at (-2, 4), passes through (-6, 7)

$r = \sqrt{(-6+2)^2 + (7-4)^2}$

$r = \sqrt{(-4)^2 + 3^2}$

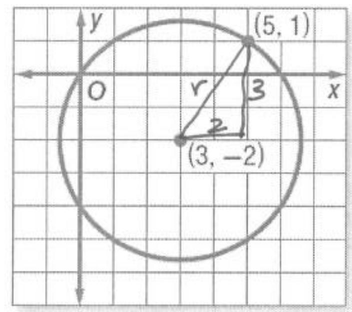
$r = \sqrt{16+9}$ $\quad r^2 = 25$

$r = \sqrt{25}$

$(x-h)^2 + (y-k)^2 = r^2$

$(x+2)^2 + (y-4)^2 = 25$

b. the circle graphed at the right



$(x-h)^2 + (y-k)^2 = r^2$

$(x-3)^2 + (y+2)^2 = 13$

c. center at (5, 4), passes through (-3, 4)

$r = \sqrt{(-3-5)^2 + (4-4)^2}$ $\quad r^2 = 64$

$r = \sqrt{(-8)^2 + 0^2}$

$r = \sqrt{64}$

$(x-h)^2 + (y-k)^2 = r^2$

$(x-5)^2 + (y-4)^2 = 64$

$r^2 = 2^2 + 3^2$

$r^2 = 4 + 9$

$r^2 = 13$

d. center at (-3, -5), passes through (0, 0)

$(x-h)^2 + (y-k)^2 = r^2$

$(x+3)^2 + (y+5)^2 = 34$

$r = \sqrt{(-3-0)^2 + (-5-0)^2}$

$r = \sqrt{(-3)^2 + (-5)^2}$ $\quad r^2 = 34$

$r = \sqrt{9+25}$

$r = \sqrt{34}$

e. The equation of a circle is $(x-4)^2 + (y+1)^2 = 9$. State the center and the radius. Then graph the equation.

$(x-h)^2 + (y-k)^2 = r^2$

center: $(4, -1)$ $\quad r^2 = 9$

$r = \pm\sqrt{9}$

$r = 3$

✱ f. For each circle with the given equation, state the coordinates of the center and the measure of the radius. Then graph the equation.

EQUATIONS OF CIRCLES (GENERAL FORM) [10]

a. Express $(x-5)^2$ as a trinomial.

$(x-5)(x-5)$

$x^2-10x+25$

b. Express $(x+4)^2$ as a trinomial.

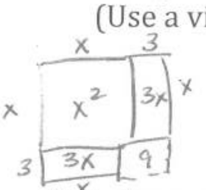
$(x+4)(x+4)$

$x^2+8x+16$

c. Factor the trinomial: $x^2+12x+36$.

$(x+6)^2$

d. Complete the square to solve the following equation: $x^2+6x-40=0$
(Use a visual representation to support your work).



$x^2+6x=40$

$x^2+6x+9=49$

$\sqrt{(x+3)^2}=\sqrt{49}$

$\{-10,4\}$

$x+3=\pm 7$

$x=-3\pm 7$

$x=-3+7$

$x=4$

$x=-3-7$

$x=-10$

Sometimes equations of circles are presented in simplified form. To easily identify the center and the radius of the circle, we sometimes need to factor and/or complete the square in order to rewrite the equation in its center-radius or standard form.

1. Rewrite the following equations in the form $(x-h)^2+(y-k)^2=r^2$.
   a. $x^2+4x+4+y^2-6y+9=36$

   $(x+2)^2+(y-3)^2=36$

   b. $x^2-10x+25+y^2+14y+49=4$

   $(x-5)^2+(y+7)^2=4$

2. What is the center and radius of the following circle?
   $x^2+4x+y^2-12y-41=0$

   $x^2+4x\underbrace{+4}+y^2-12y\underbrace{+36}=41+4+36$

   $(x+2)^2+(y-6)^2=81$

   center $(-2,6)$

   radius $=\sqrt{81}=9$

3. Identify the center and radius for each of the following circle.
   a. $x^2 - 20x + y^2 + 6y = 35$

$x^2 - 20x \underbrace{+100} + y^2 + 6y \underbrace{+9} = 35 + 100 + 9$

$(x-10)^2 + (y+3)^2 = 144$ center: $(10, -3)$ radius $= \sqrt{144} = 12$

   b. $x^2 - 3x + y^2 - 5y = \frac{19}{2}$

$x^2 - 3x + \frac{9}{4} + y^2 - 5y \underbrace{+\frac{25}{4}} = \frac{19}{2} + \frac{9}{4} + \frac{25}{4}$

$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 = 18$ center: $\left(\frac{3}{2}, \frac{5}{2}\right)$ radius $= \sqrt{18}$ $= 3\sqrt{2}$

4. Could the circle with equation $x^2 - 6x + y^2 - 7 = 0$ have a radius of 4? Why or why not?

$x^2 - 6x \underbrace{+9} + y^2 = 7 + 9$

$(x-3)^2 + y^2 = 16$

yes. The center of the circle is $(3,0)$ and the radius is $\sqrt{16} = 4$.

5. Stella says the equation $x^2 - 8x + y^2 + 2y = 5$ has a center of (4, -1) and a radius of 5. Is she correct? Why or why not?

$x^2 - 8x \underbrace{+16} + y^2 + 2y \underbrace{+1} = 5 + 16 + 1$

$(x-4)^2 + (y+1)^2 = 22$

she is correct that the center is $(4,-1)$ however, the radius is not 5. The radius is $\sqrt{22}$.

6. Identify the graphs of the following equations as a circle, a point, or an empty set.
   a. $x^2 + y^2 + 4x = 0$

$x^2 + 4x \underbrace{+4} + y^2 = 0 + 4$

$(x+2)^2 + y^2 = 4$ circle center: $(-2, 0)$ $r = \sqrt{4} = 2$

   b. $x^2 + y^2 + 6x - 4y + 15 = 0$

$x^2 + 6x \underbrace{+9} + y^2 - 4y \underbrace{+4} = -15 + 9 + 4$

$(x+3)^2 + (y-2)^2 = -2$ empty set

   c. $2x^2 + 2y^2 - 5x + y + \frac{13}{4} = 0$

$2x^2 - 5x \underline{\quad} + 2y^2 + y \underline{\quad} = -\frac{13}{4}$

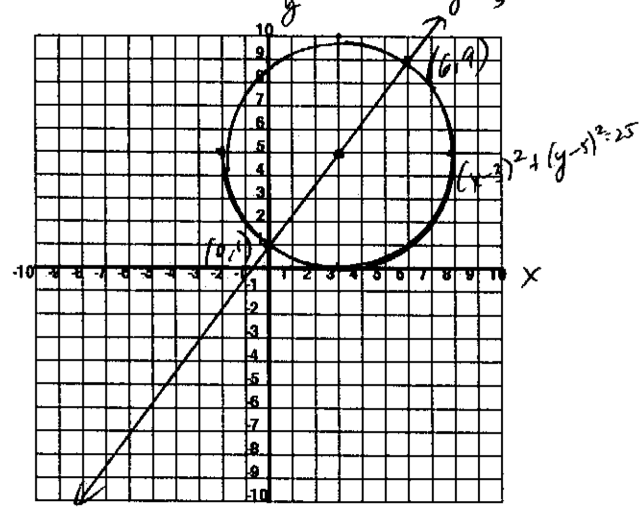
$x^2 - \frac{5}{2}x \underbrace{+\frac{25}{16}} + y^2 + \frac{1y}{2} \underbrace{+\frac{1}{16}} = \frac{-13}{4} + \frac{25}{16} + \frac{1}{16}$

$\left(x - \frac{5}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{-13}{8}$ empty set

EQUATIONS FOR TANGENT LINES [11]

1. Consider the circle with equation $(x-3)^2 + (y-5)^2 = 25$. Find the equations of two tangent lines to the circle that each have a slope of $\frac{-3}{4}$.

$(3,5)$ $r=\sqrt{25}$
$r=5$



Diameter

$\perp m = \frac{4}{3}$ $(3,5)$

$y-5=\frac{4}{3}(x-3)$

$y-5=\frac{4}{3}x-4$

$y=\frac{4}{3}x+1$

$(x-3)^2+(y-5)^2=25$

$(x-3)(x-3) + (\frac{4}{3}x+1-5)^2=25$

$x^2-6x+9 + (\frac{4}{3}x-4)(\frac{4}{3}x-4)=25$

$x^2-6x+9 + \frac{16}{9}x^2 - \frac{32}{3}x+16=25$

$\frac{25}{9}x^2 - \frac{50}{3}x+25=25$

$9\left(\frac{25}{9}x^2 - \frac{50}{3}x = 0\right)$

$25x^2-150x=0$

$25x(x-6)=0$

$x=0$ $x=6$

$y=\frac{4}{3}(0)+1$ $y=\frac{4}{3}(6)+1$

$y=1$ $y=9$

$(0,1)$ $(6,9)$

tangent lines

$m=\frac{-3}{4}$ $(0,1)$

$y-1=\frac{-3}{4}(x-0)$

$y=\frac{-3}{4}x+1$

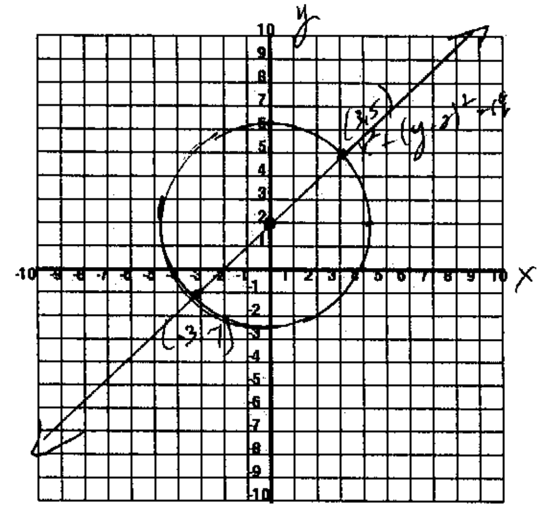
$y-9=\frac{-3}{4}(x-6)$

or

$y-9=\frac{-3}{4}x+\frac{18}{4}$

$y=\frac{-3}{4}x+\frac{27}{2}$

2. Consider the circle with equation $x^2 + (y-2)^2 = 18$. Find the equations of two tangent lines to the circle that each have a slope of -1.



$(0,2)$ $\quad r = \sqrt{18}$

$r = 3\sqrt{2}$

≈ 4.24

diameter

$\perp m = 1 \quad (0,2)$

$y = x + 2$

$(x^2) + (y-2)^2 = 18$

$x^2 + (x+2-2)^2 = 18$

$x^2 + x^2 = 18$

$2x^2 = 18$

$x^2 = 9$

$x = \pm 3$

$y = -3 + 2 \qquad y = 3 + 2$

$y = -1 \qquad y = 5$

$(-3,-1) \qquad (3,5)$

tangents

$m = -1 \quad (-3,-1)$

$y + 1 = -1(x+3)$

or

$y + 1 = -x - 3$

$y = -x - 4$

$m = -1 \quad (3,5)$

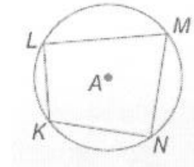
$y - 5 = -1(x-3)$

or

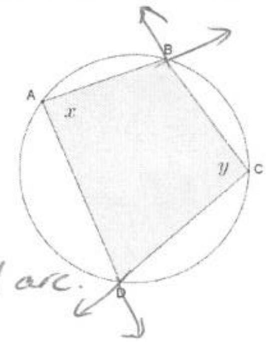
$y - 5 = -x + 3$

$y = -x + 8$

CYCLIC QUADRILATERALS [12]

Theorem		
Words	*Example*	*Figure*
If a quadrilateral is inscribed in a circle (cyclic quadrilateral), then its opposite angles are supplementary.	$m\angle L + m\angle N = 180^\circ$ $m\angle K + m\angle M = 180^\circ$	

Given: Cyclic quadrilateral *ABCD*
Prove: $x + y = 180°$



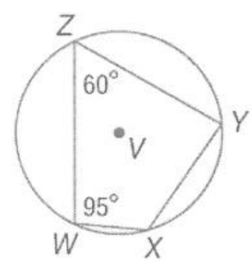
STATEMENTS	REASONS
① cyclic quad ABCD	① Given
② $x = \frac{1}{2} m\widehat{BCD}$ $y = \frac{1}{2} m\widehat{BAD}$	② An inscribed ∡ of a circle equals half the measure of its intercepted arc.
③ $x + y = \frac{1}{2} m\widehat{BCD} + \frac{1}{2} m\widehat{BAD}$	③ Addition prop.
④ $x + y = \frac{1}{2}(m\widehat{BCD} + m\widehat{BAD})$	④ Factoring GCF
⑤ $m\widehat{BCD} + m\widehat{BAD} = 360^\circ$	⑤ The sum of the arcs of a circle is 360°
⑥ $x + y = \frac{1}{2}(360^\circ)$	⑥ substitution
⑦ $x + y = 180^\circ$	⑦ simplification

Examples

a. Quadrilateral *WXYZ* is inscribed in circle *V*. Find $m\angle X$ and $m\angle Y$.

$m\angle X + 60^\circ = 180$ $m\angle Y + 95^\circ = 180^\circ$

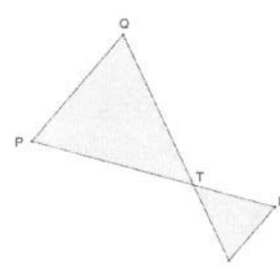
$m\angle X = 120^\circ$ $m\angle Y = 85^\circ$



b. Quadrilateral *PQRS* is a cyclic quadrilateral. Explain why $\Delta PQT \sim \Delta SRT$.

$\angle Q \cong \angle R$ In a circle, if 2 inscribed ∡'s share the same intercepted arc, then they are ≅.

$\angle P \cong \angle S$

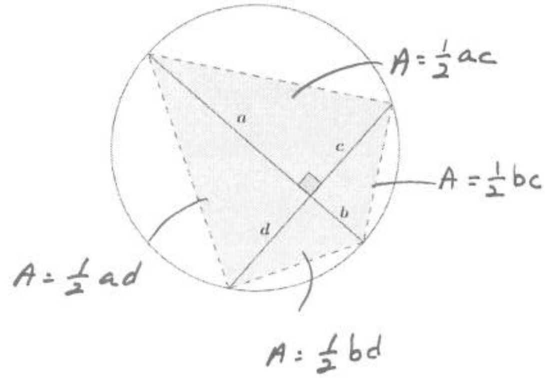


$\Delta PQT \sim \Delta SRT$ by AA Similarity

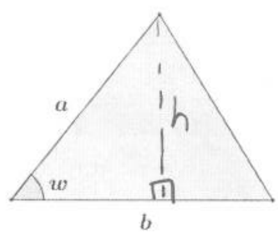
c. A cyclic quadrilateral has perpendicular diagonals. What is the area of the quadrilateral in terms of a, b, c, and d as shown?

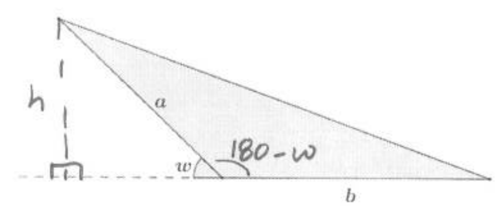
$A = \frac{1}{2}ac + \frac{1}{2}bc + \frac{1}{2}bd + \frac{1}{2}ad$

$A = \frac{1}{2}(ac + bc + bd + ad)$



d. Show that the triangle in the diagrams below both have the same area $\left(\frac{1}{2}absin(w)\right)$.





$\sin w = \frac{h}{a}$

$h = a \sin w$

$A = \frac{1}{2}bh$

$A = \frac{1}{2}b\,a \sin w$

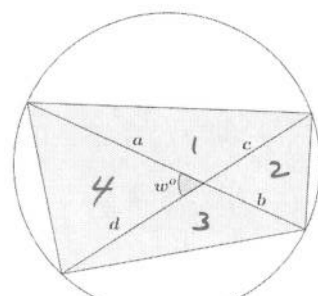
$\sin w = \frac{h}{a}$

$h = a \sin w$

$A = \frac{1}{2}bh$

$A = \frac{1}{2}b\,a \sin w$

e. Show that the area of the cyclic quadrilateral shown in the diagram below is $Area = \frac{1}{2}(a+b)(c+d)\sin(w)$.



$A_{\Delta 1} = \frac{1}{2}ac \sin(180 - w)$

$A_{\Delta 2} = \frac{1}{2}bc \sin w$

$A_{\Delta 3} = \frac{1}{2}bd \sin(180 - w)$

$A_{\Delta 4} = \frac{1}{2}ad \sin w$

$A = \frac{1}{2}ac\sin(180-w) + \frac{1}{2}bc\sin w + \frac{1}{2}bd\sin(180-w) + \frac{1}{2}ad\sin w$

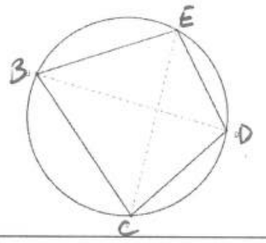
$= \frac{1}{2}ac\sin w + \frac{1}{2}bc\sin w + \frac{1}{2}bd\sin w + \frac{1}{2}ad\sin w$

$= \frac{1}{2}\sin w\,(ac + bc + bd + ad)$

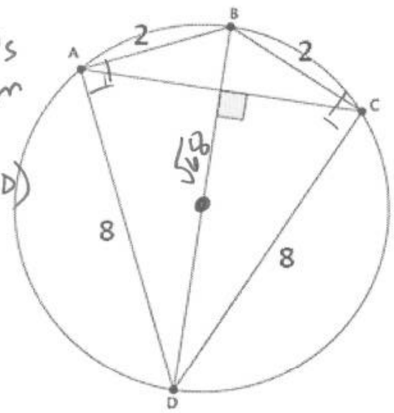
$= \frac{1}{2}\sin w\,(a+b)(c+d)$

PTOLEMY'S THEOREM [13]

http://tube.geogebra.org/student/m34477

Ptolemy's Theorem		
Words	*Example*	*Figure*
In a cyclic quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.	$BD \cdot CE = DE(BC) + BE(CD)$	

1. What is the length of the chord $\overline{AC}$? Explain your answer.



$(BD)^2 = (AB)^2 + (AD)^2$

$(BD)^2 = 2^2 + 8^2$

$(BD)^2 = 4 + 64$

$(BD)^2 = 68$

$BD = \sqrt{68}$

Ptolemy's theorem

$AC(BD) = AB(CD) + BC(AD)$

$AC(\sqrt{68}) = 2(8) + 2(8)$

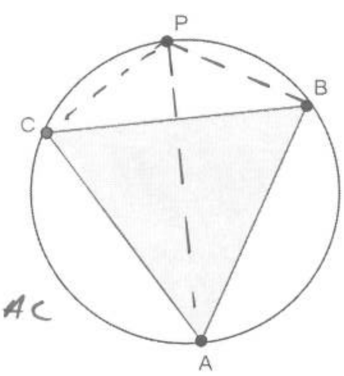
$(AC)\sqrt{68} = 32$

$AC = \frac{32}{\sqrt{68}} = \frac{32}{2\sqrt{17}}$

$AC = \frac{16}{\sqrt{17}} = \frac{16\sqrt{17}}{17}$

An inscribed ∡ of a circle that intercepts a semicircle is a rt. ∡.
Therefore $\triangle ABD$ is a right $\triangle$.

2. An equilateral triangle is inscribed in a circle. If P is a point on the circle, what does Ptolemy's theorem have to say about the distances from this point to the three vertices of the triangle?



$PA(BC) = PC(BA) + BP(AC)$

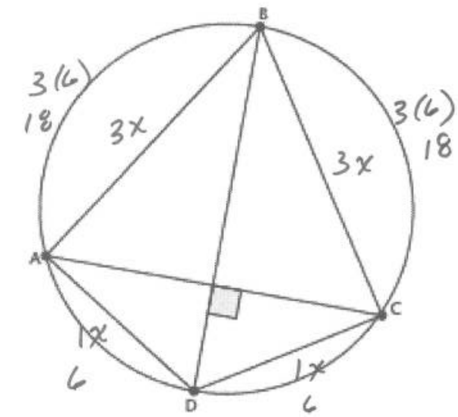
Since $\triangle ABC$ is equilateral, $AB = BC = AC$

so

$PA = PC + PB$

the sum of the two shorter distances equals the longer distance.

3. Kite $ABCD$ is inscribed in a circle. The kite has an area of 108 sq. in., and the ratio of the lengths of the non-congruent adjacent sides is 3 : 1. What is the perimeter of the kite?



$A = \frac{1}{2}(AC)(BD)$

$108 = \frac{1}{2}(AC)(BD)$

$216 = (AC)(BD)$

by Ptolemy's theorem

$AC \cdot BD = AB(CD) + BC(AD)$

$216 = 3x(x) + 3x(x)$

$216 = 3x^2 + 3x^2$

$216 = 6x^2$

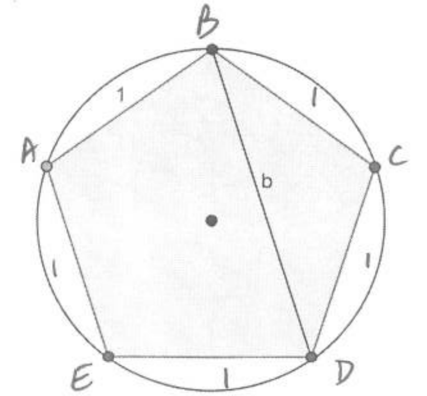
$x^2 = 36$

$x = 6$

$P = 18 + 18 + 6 + 6$

$P = 48$ inches

4. Draw a regular pentagon of side length 1 in a circle. Let b be the length of its diagonals. What does Ptolemy's theorem say about the quadrilateral formed by four of the vertices of the pentagon?



$BE(AD) = BD(AE) + AB(ED)$

$b(b) = b(1) + 1(1)$

$b^2 = b + 1$

$b^2 - b - 1 = 0$

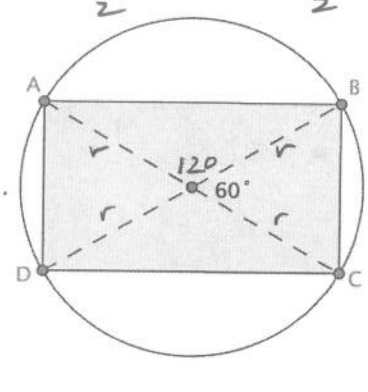
$b = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$b = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)}$

$b = \frac{1 \pm \sqrt{1+4}}{2}$

$b = \frac{1 \pm \sqrt{5}}{2}$ $\qquad \frac{1+\sqrt{5}}{2} = b$

~~5~~. The area of the inscribed quadrilateral is $\sqrt{300}$ mm². Determine the circumference of the circle.



$AD \cdot AB = \sqrt{300}$ since $ABCD$ is a rectangle.

$\sqrt{2r^2(1-\cos 60°)}\left(\sqrt{2r^2(1-\cos 120°)}\right) = \sqrt{300}$

$AB^2 = r^2 + r^2 - 2rr\cos 120°$

$(AB)^2 = 2r^2 - 2r^2\cos 120°$

$(AB)^2 = 2r^2(1 - \cos 120°)$ $\qquad AB = \sqrt{2r^2(1-\cos 120°)}$

$\sqrt{r^2}\,\sqrt{3r^2} = \sqrt{300}$

$\sqrt{3r^4} = \sqrt{300}$

$3r^4 = 300$

$r^4 = 100$

$r = \sqrt{10}$

$(AD)^2 = r^2 + r^2 - 2rr\cos 60°$

$(AD)^2 = 2r^2 - 2r^2\cos 60°$

$(AD)^2 = 2r^2(1 - \cos 60°)$

$AD = \sqrt{2r^2(1-\cos 60°)}$